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The building volume with hygroscopic materials—an analytical study of a classical building physics problem

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Abstract

A global analytical solution covering all cases of a building volume with hygroscopic materials is given. The mathematical and physical simplifications and assumptions are quite modest. Isothermality is not assumed. Examples are rooms, attics, subfloor spaces and building cavities. All share the same physics describing the vapour pressure in the building volume and the moisture content in the hygroscopic materials as a function of building volume temperature and moisture emission rates, external vapour pressure and building volume ventilation levels, heat and mass transfer between the building volume and the hygroscopic materials, and heat and mass storage and transfer within the hygroscopic materials. © 2002 Elsevier Science Ltd. All rights reserved.

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1. Introduction

A building volume containing hygroscopic materials (BVHM) is a central concept in building physics. Many different parts of a building are of this nature, e.g., an occupied space [1–4] where the hygroscopic materials are the surrounding floors, ceiling and walls, fabrics in furniture, bedding, carpets, drapes, etc., a wall cavity [5] where the surrounding hygroscopic materials may be the indoor lining, the outdoor cladding and structural timbers, an attic cavity [6] where the surrounding hygroscopic materials may be the roof cladding, the ceiling lining, insulation materials and structural timbers, a roof cavity in a flat or cathedral roof [7] where the surrounding hygroscopic materials may be the roof cladding, the ceiling lining, insulation materials and structural timbers, a subfloor cavity [8] where the surrounding hygroscopic materials may be the floor, foundations, the ground if not covered. These building volumes may have moisture sources, people and their activities in the case of an occupied space, open water tanks in attics, water evaporating from the ground if not covered in the case of a subfloor cavity, plumbing leaks in wall cavities, etc. Each also will have air exchanges into and out of the building volume.

There are now many numerical models that are capable of describing heat and moisture transfer within the building envelope, some of them very sophisticated [9]. Generally, these models are centred on the performance of the various layers in the envelope and are not usually concerned with conditions in a building volume. However, some models exist where a room is the building volume and the envelope is the hygroscopic material [1–4]. These models do not seek a global understanding of all BVHM cases, but rather give numerical results on a case by case basis, often to be used for envelope design.

Nevertheless, there are a number of key physical processes involved in all BVHM cases, viz., ventilation or air infiltration into and out of the building volume, release of moisture from sources within the volume, moisture transfer between the building volume and the hygroscopic materials, moisture transport within the hygroscopic materials, and moisture storage within the hygroscopic materials. The physical parameters governing these BVHM process are, respectively, ventilation rate and external climate, moisture emission rates, surface heat and mass transfer coefficients, heat and moisture diffusivities, and the sorption properties of the hygroscopic materials.

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Nomenclature

A	area of equivalent hygroscopic volume participating in heat and moisture transfer with the building volume (m^2)
c	moisture concentration (kg m^{-3})
c_i	moisture content of the building volume air (kg m^{-3})
c_o	moisture content of external air (kg m^{-3})
d_T	effective penetration depth for heat transfer (m) (Eq. (17))
D_p	coefficient of moisture diffusion under vapour pressure driving forces (s)
F	air change rate (s^{-1})
H	surface heat transfer coefficient (W m^{-2} °C^{-1})
$\Im$	imaginary part of
j	unit imaginary number
k	thermal conductivity (W m^{-1} °C^{-1})
$\mathscr{L}^{-1}$	inverse Laplace Transform
m	moisture content (kg m^{-3})
p	vapour pressure (Pa)
p_h	vapour pressure of the equivalent hygroscopic volume (Pa)
p_i	vapour pressure of the building volume air (Pa)
$p_{initial}$	initial equilibrium vapour pressure of equivalent hygroscopic volume and building volume air (Pa)
p_s	vapour pressure at the surface of the equivalent hygroscopic volume (Pa)
p_{sat}	saturated vapour pressure (Pa)
r	vapour resistance for mass transfer between the equivalent hygroscopic volume and the building volume air (N s kg^{-1})
R	thermal resistance for heat transfer between the equivalent hygroscopic volume and the building volume air (°C m^2 W^{-1})
S	moisture emission rate (kg s^{-1})
T	temperature (°C)
T_i	temperature of the building volume air (°C)
$T_{initial}$	initial equilibrium temperature of equivalent hygroscopic volume and building volume air (°C)
T_s	temperature at the surface of the equivalent hygroscopic volume (Pa)

t	time (s)
t_h	time constant associated with the equivalent hygroscopic volume (s) (Eq. (31))
t_p	time constant for moisture transfer (s) (Eq. (32))
t_T	time constant for heat transfer (s) Eq. (14)
V	volume of the building volume (m^3)
W	molecular weight of water, 18 kg kmol^{-1}
Δm	moisture content of equivalent hygroscopic volume in excess of its equilibrium value (kg m^{-3})
Δp_h	pressure of equivalent hygroscopic volume in excess of the equilibrium value, $p_{initial}$ (Pa)
Δp_i	pressure of building volume in excess of the equilibrium value, $p_{initial}$ (Pa)
Δp_s	pressure of the surface of the equivalent hygroscopic volume in excess of the equilibrium value, $p_{initial}$ (Pa)
ΔT_h	temperature of equivalent hygroscopic volume in excess of the equilibrium value, $T_{initial}$ (°C)
ΔT_i	temperature of building volume in excess of the equilibrium value, $T_{initial}$ (°C)
ΔT_s	temperature of the surface of the equivalent hygroscopic volume in excess of the equilibrium value, $T_{initial}$ (°C)

Greek letters

α, β, γ	roots of a cubic defined by Eq. (33) (s$^{-1/2}$)
α', β'	inverse of exponential time constants (s^{-1})
κ	diffusivity (m^2 s^{-1})
ρ	density (kg m^{-3})
δ, ϕ, θ	phase angle (rad)
ϕ	relative humidity as a fraction
ν	related to the ideal gas law and defined by Eq. (20)
ω	angular frequency (rad s^{-1})

Since all BVHM cases have the same physics it should be possible to characterise them with one global solution, despite the myriad of different specific cases. This paper derives a global solution using only moderate physical and mathematical assumptions and simplifications. The author has explored this physics in some simplified cases [10,11] in which the hygroscopic materials have been lumped into one node. In Ref. [10] isothermality was assumed, and in Ref. [11] the interest was in sinusoidal excitations only.

The assumption of isothermality within the hygroscopic materials is not made in this work. It is possible to provide approximate solutions to BVHM cases by assuming isothermality, but doing so causes a loss in information concerning the important difference in the way in which change of temperature affects the moisture in the building volume air from how change in temperature affects the moisture in the hygroscopic materials. The immediate effect of an increase in temperature in the building volume air is to leave its vapour pressure unchanged but to lower its relative humidity. (Any subsequent change in building volume vapour pressure is due to the manner in which the building volume air then reacts with the hygroscopic materials.) On the other hand, the immediate effect of an increase in temperature in the hygroscopic materials is to leave their internal relative humidities unchanged but to increase their vapour pressures. This is so because a change in temperature does not cause an immediate change in moisture content, and thus, through the sorption curves for the hygroscopic materials, does not cause an immediate change in relative humidity. (Of course, once the hygroscopic materials are no longer in internal equilibrium or in equilibrium with the building volume, moisture transfer processes take place that do begin to change hygroscopic material internal moisture contents and relative humidities.)

2. Physical model

We consider in this work a very general building volume containing hygroscopic materials. The building volume, which may be a room, an attic, a subfloor space, a wall or ceiling cavity, etc., contains or is contained by hygroscopic materials, such as timber, wall, ceiling or floor linings, external claddings, carpets, drapes, furniture, bedding, etc. Within the building volume there may be a moisture source such as moisture from people or their activities, evaporating soil moisture, open water or fish tanks, plants, etc. The building volume exchanges air to and from regions outside the volume, perhaps the outdoors, another room, another building cavity, etc.

To describe this BVHM we write the differential equations governing heat and moisture transfer within the hygroscopic materials, heat and moisture transfer between the hygroscopic materials and the building volume air, and the differential equation describing moisture content of the building volume air as a function of ventilation, moisture release rates, and interaction with hygroscopic materials. Heat transfer is treated as uncoupled from moisture processes. Latent heat effects and temperature-dependant thermal conductivities are ignored.

In this model the building volume is lumped as one unit, i.e. internal heat and moisture transfer by convection and radiation is not accounted for. The hygroscopic materials are collected into one equivalent volume with properties representative of the average hygroscopic properties of all the hygroscopic materials. This will be called the *equivalent hygroscopic volume* throughout this paper. It is important to note that the building envelope alone is part of, but not the totality of, the equivalent hygroscopic volume. This equivalent hygroscopic volume is taken of thickness much greater than the effective penetration depth, $d = \sqrt{D_p/(2\omega \partial m/\partial p)}$ [12], associated with the period $T = 2\pi/\omega$ of typical fluctuations in the temperature and vapour pressure of the building volume air. In practice, the building volume is most likely to display daily and yearly fluctuations in its psychrometric conditions. For hygroscopic building materials the mass transfer effective penetration depth is of the order of millimetres for daily fluctuations and of the order of centimetres for yearly fluctuations. Specifying that the equivalent hygroscopic volume is much thicker than these values is to state that there will be no significant fluctuations in moisture content at depths greater than a few millimetres if daily fluctuations in moisture content are under consideration, or a few centimetres if yearly fluctuations in moisture content are under consideration. If the equivalent hygroscopic volume is at least this thick then mathematically it can be taken to be of infinite thickness.

The same assumption is made for the thickness of the equivalent hygroscopic volume with respect to temperature fluctuations. For building materials the effective depth is of the order of centimetres for daily temperature fluctuations and of meters for yearly fluctuations.

Moisture transfer within the equivalent hygroscopic volume and across the boundary of the volume to the building volume is assumed to be driven by vapour pressure differences. The sorption isotherm of the hygroscopic material is taken as constant over the range of humidities it is subjected to.

Within the equivalent hygroscopic volume, taking these assumptions into account, heat and moisture transfer is governed by the differential equations:

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}, \tag{1}$$

where κ is the thermal diffusivity given by

$$\kappa = \frac{k}{\rho c}$$

for heat transfer and

$$\frac{\partial m}{\partial t} = D_p \frac{\partial^2 p_h}{\partial x^2} \tag{2}$$

for moisture transfer.

Heat and moisture transfer across the surface of the equivalent hygroscopic volume is given by

$$\frac{T_i - T_s}{R} = -k \left. \frac{\partial T}{\partial x} \right|_s \tag{3}$$

and

$$\frac{p_i - p_s}{r} = -D_p \left.\frac{\partial p}{\partial x}\right|_s, \tag{4}$$

respectively, where the subscript i stands for the building volume, and subscript s stands for the surface of the equivalent hygroscopic volume.

The moisture balance of the building volume air is

$$V\frac{dc_i}{dt} = VF(c_0 - c_i) + \frac{A(p_s - p_i)}{r} + S, \tag{5}$$

where c is vapour concentration in units of kg m^{-3} and the subscript o is for the air external to the building volume that is being exchanged with the building volume air.

3. Model solutions—temperature

3.1. Temperature transfer function and impulse response

For the purposes of this work, building volume conditions, especially vapour pressure, are of atmost interest. We begin by solving for the temperature in the equivalent hygroscopic volume. Initially, the equivalent hygroscopic volume is assumed to be in thermal equilibrium with the building volume air at temperature $T_{initial}$. We denote by ΔT_h the equivalent hygroscopic volume temperature in excess of this initial temperature i.e.

$$\Delta T_h = T_h - T_{initial}. \tag{6}$$

Eqs. (1) and (3) become

$$\frac{\partial \Delta T_h}{\partial t} = \kappa \frac{\partial^2 \Delta T_h}{\partial x^2}, \tag{7}$$

$$\frac{\Delta T_i - \Delta T_s}{R} = -k \left.\frac{\partial \Delta T_h}{\partial x}\right|_s. \tag{8}$$

Using a Laplace Transform approach to finding the solution to these equations, we first transform Eqs. (7) and (8) to

$$s\Delta \mathbf{T}_h - \Delta T_h(t=0) = \kappa \frac{\partial^2 \Delta \mathbf{T}_h}{\partial x^2}, \tag{9}$$

$$\frac{\Delta \mathbf{T}_i - \Delta \mathbf{T}_s}{R} = -k \left.\frac{\partial \Delta \mathbf{T}_h}{\partial x}\right|_s, \tag{10}$$

$\Delta T_h(t=0)$ is zero since the equivalent hygroscopic volume being taken as is in equilibrium with the building volume initially.

It is straightforward to solve the differential equation, Eq. (9) with the boundary condition given by Eq. (10) and the boundary condition that $\Delta \mathbf{T}_h$ will be zero at $x = \infty$, the solution for the equivalent hygroscopic volume temperature being

$$\Delta \mathbf{T}_h = \frac{\Delta \mathbf{T}_i \exp(-x\sqrt{s/\kappa})}{1 + kR\sqrt{s/\kappa}}. \tag{11}$$

If the building volume temperature T_i is an impulse of temperature of unit magnitude, then Eq. (11) with $\Delta \mathbf{T}_i = 1$ is the transfer function, $\mathbf{D}_h$, describing the behaviour of the equivalent hygroscopic volume, i.e.

$$\mathbf{D}_h = \frac{\exp(-x\sqrt{s/\kappa})}{1 + kR\sqrt{s/\kappa}}. \tag{12}$$

Given this transfer function well-known solutions can be given, e.g., the response of the equivalent hygroscopic volume to a pulse of amplitude ΔT_i which is

$$\Delta T_h = \frac{\Delta T_i}{t_T} \left\{ \sqrt{\frac{t_T}{\pi t}} \exp\left(-\frac{x^2}{4\kappa t}\right) \right.$$
$$\left. - \exp\left(\frac{x}{\sqrt{\kappa t_T}} + \frac{t}{t_T}\right) \text{erfc}\left(\frac{x}{2\sqrt{\kappa t}} + \sqrt{\frac{t}{t_T}}\right) \right\}, \tag{13}$$

where

$$t_T = \frac{(kR)^2}{\kappa} \tag{14}$$

and where $\text{erfc}(x)$ is the complementary error function.

Solutions under other conditions can be found by convolution, e.g. the hygroscopic volume response to a step function is

$$\Delta T_h = \Delta T_i \left\{ \text{erfc}\left(\frac{x}{2\sqrt{\kappa t}}\right) \right.$$
$$\left. - \exp\left(\frac{x}{\sqrt{\kappa t_T}} + \frac{t}{t_T}\right) \text{erfc}\left(\frac{x}{2\sqrt{\kappa t}} + \sqrt{\frac{t}{t_T}}\right) \right\} \tag{15}$$

while the response of the equivalent hygroscopic volume to a sinusoidal fluctuation of building volume temperature is

$$\Delta T_h = \Delta T_i \frac{1}{\sqrt{(1 + \sqrt{\omega t_T})^2 + \sqrt{\omega t_T}}}$$
$$\times \exp(j(\omega t - x/(2d_T) - \delta + \phi)), \tag{16}$$

where d_T is the effective penetration depth for heat transfer, given by

$$d_T = \sqrt{\frac{\kappa}{2\omega}} \tag{17}$$

and where δ is

$$\delta = \tan^{-1}\left(\frac{\sqrt{\omega t_T}}{1 + \sqrt{\omega t_T}}\right). \tag{18}$$

4. Model solutions—vapour pressure

4.1. Vapour pressure transfer function and impulse response

By putting $\delta c_i/\partial t$ equal to zero in Eq. (5) the well-known result is derived that at equilibrium the initial building volume vapour pressure is given by

$$p_{initial} = p_0 + \frac{S}{vF}, \tag{19}$$

where v is related to the ideal gas law and defined by

$$v = \frac{VW}{RT}. \tag{20}$$

At initial equilibrium the initial vapour pressures in the equivalent hygroscopic volume will also be equal $p_{initial}$. As for temperatures, vapour pressures in excess of the equilibrium values will now be defined as

$$\Delta p_h = p_h - p_{initial}, \quad \Delta p_i = p_i - p_{initial},$$

$$\Delta p_o = p_o - p_{initial}.$$

With these definitions Eqs. (2), (4), and (5) become, respectively,

$$\frac{\partial \Delta m}{\partial t} = D_p \frac{\partial^2 \Delta p_h}{\partial x^2}, \tag{21}$$

$$\frac{\Delta p_i - \Delta p_s}{r} = -D_p \frac{\partial \Delta p}{\partial x}\bigg|_s, \tag{22}$$

$$V \frac{d\Delta c_i}{dt} = VF(\Delta c_o - \Delta c_i) + \frac{A(\Delta p_s - \Delta p_i)}{r} + S. \tag{23}$$

To calculate vapour pressures we first transform Eq. (21) to

$$\frac{\partial \Delta m}{\partial t} = \frac{\partial \Delta m}{\partial p}\frac{\partial \Delta p_h}{\partial t} + \frac{\partial \Delta m}{\partial T}\frac{\partial \Delta T_h}{\partial t} = D_p \frac{\partial^2 \Delta p_h}{\partial x^2}. \tag{24}$$

For notational simplicity, we will write $\partial \Delta m/\partial p$ and $\partial \Delta m/\partial T$ as $\partial m/\partial p$ and $\partial m/\partial T$, so that Eq. (24) will be written as

$$\frac{\partial \Delta m}{\partial t} = \frac{\partial m}{\partial p}\frac{\partial \Delta p_h}{\partial t} + \frac{\partial m}{\partial T}\frac{\partial \Delta T_h}{\partial t} = D_p \frac{\partial^2 \Delta p_h}{\partial x^2}.$$

The Laplace Transform of this equation is

$$\frac{\partial m}{\partial p}(s\Delta P_h - \Delta p_h(t=0)) + \frac{\partial m}{\partial T}(s\Delta T_h - \Delta T_h(t=0))$$

$$= D_p \frac{\partial^2 \Delta P_h}{\partial x^2},$$

$\Delta p_h(t=0)$ and $\Delta T_h(t=0)$ are both zero since the equivalent hygroscopic volume is in equilibrium with the building volume initially, so we have

$$\frac{\partial m}{\partial p}s\Delta P_h + \frac{\partial m}{\partial T}s\Delta T_h = D_p \frac{\partial^2 \Delta P_h}{\partial x^2}. \tag{25}$$

Moisture transport between the building volume and the equivalent hygroscopic volume is governed by Eq. (22) which transforms to

$$\frac{\Delta P_i - \Delta P_s}{r} = -D_p \frac{\partial \Delta P}{\partial x}\bigg|_s. \tag{26}$$

Eq. (23) governing the moisture balance of the building volume is put into terms of vapour pressure only by using the ideal gas law in the form of

$$p = c \frac{RT}{W}.$$

Eq. (23) becomes

$$v \frac{d\Delta p_i}{dt} + \left(vF + \frac{A}{r}\right)\Delta p_i = vF\Delta p_o + \frac{A}{r}\Delta p_s + S. \tag{27}$$

The Laplace Transform of Eq. (27) is

$$\left(vs + vF + \frac{A}{r}\right)\Delta P_i - v\Delta p_i(t=0)$$

$$= vF\Delta P_s + \frac{A}{r}\Delta P_s + S,$$

$\Delta p_i(t=0)$ is zero under the assumption of initial equilibrium, i.e.

$$\left(vs + vF + \frac{A}{r}\right)\Delta P_i = vF\Delta P_o + \frac{A}{r}\Delta P_s + S. \tag{28}$$

Since the sorption isotherms are considered constant over the range of humidities in question $\partial m/\partial p$ and $\partial m/\partial T$ can be taken as constant over the temperature and vapour pressure ranges under consideration. We have

$$\frac{\partial m}{\partial p} = \frac{\partial m}{\partial \phi}\frac{\partial \phi}{\partial p} = \frac{\partial m}{\partial \phi}\frac{\partial(p/p_{sat})}{\partial p} = \frac{1}{p_{sat}}\frac{\partial m}{\partial \phi}$$

and

$$\frac{\partial m}{\partial T} = \frac{\partial m}{\partial \phi}\frac{\partial \phi}{\partial T} = \frac{\partial m}{\partial \phi}\frac{\partial(p/p_{sat})}{\partial T} = p\frac{\partial(1/p_{sat})}{\partial T}\frac{\partial m}{\partial \phi}.$$

The above analysis shows that $\partial m/\partial p$ and $\partial m/\partial T$ consists of two parts, viz. $\partial m/\partial \phi$ that describes the sorption curve of the equivalent hygroscopic volume, and the terms $1/p_{sat}$ or $p\partial(1/p_{sat})/\partial T$ which describe the psychrometric properties of water vapour pressure. To assume that $\partial m/\partial p$ and $\partial m/\partial T$ can be taken as constant over the temperature and vapour pressure ranges under consideration is to assume that the sorption curve for the equivalent hygroscopic volume is linear over the moisture contents under consideration, and that the psychrometric properties of water vapour pressure can be approximated as constant over the temperature and vapour pressure ranges under consideration.

Standard methods are used to solve differential equations (25), (26) and (28) simultaneously with ΔP_h, ΔP_s and ΔP_i as the unknown variables and with the boundary condition $\Delta P_h = 0$ at $x = \infty$ giving for ΔP_i

$$\Delta P_i = \frac{(S/vF + \Delta P_o)(1 + \sqrt{t_p s}) - B(s)\Delta T_i}{\frac{\sqrt{t_p s^{3/2}}}{F} + \frac{s}{F} + (1 + \frac{1}{Ft_h})\sqrt{t_p s} + 1}, \tag{29}$$

where

$$B(s) = \frac{\frac{\partial m}{\partial T}\frac{A}{V}\sqrt{s}}{F(1 + \sqrt{t_T s})(\sqrt{\frac{1}{\kappa}} + \sqrt{\frac{\partial m/\partial p}{D}})}, \tag{30}$$

$$t_h = \frac{vr}{A} \tag{31}$$

and

$$t_p = \frac{\partial m}{\partial p}D_p r^2. \tag{32}$$

If we call the denominator of Eq. (29) $\mathscr{D}(s)$, i.e.

$$\mathscr{D}(s) = \frac{\sqrt{t_p}\,s^{3/2}}{F} + \frac{s}{F} + \left(1 + \frac{1}{Ft_h}\right)\sqrt{t_p}\,s + 1$$

$$= \frac{\sqrt{t_p}}{F}(\sqrt{s} - \alpha)(\sqrt{s} - \beta)(\sqrt{s} - \gamma), \tag{33}$$

then the transfer functions for a unit pulse of moisture source S is given by putting $S = 1$ and putting ΔP_o and ΔT_i to zero in Eq. (29) to give the Laplace Transform of a unit pulse of moisture, viz.

$$\Delta P_i = \frac{(1 + \sqrt{t_p s})}{vF\mathscr{D}(s)}. \tag{34}$$

Likewise the transfer function for unit pulses of ΔP_o and ΔT_i are, respectively,

$$\Delta P_i = \frac{(1 + \sqrt{t_p s})}{\mathscr{D}(s)} \tag{35}$$

and

$$\Delta P_i = \frac{-B(s)}{\mathscr{D}(s)}. \tag{36}$$

The inverse Laplace Transforms on Eqs. (34)–(36) can be found by standard methods [13] to give for the response of the building volume air to a pulse of moisture of amplitude S, giving

$$\Delta p_i = \frac{S/v}{\sqrt{t_p}}\left\{\frac{\alpha E(\alpha, t)}{|\bar\beta - \alpha|^2} - \frac{2}{\Im(\beta)/j}\left(\frac{\beta E(\beta, t)}{\beta - \alpha} - \frac{\bar\beta E(\bar\beta, t)t}{\bar\beta - \alpha}\right)\right\}$$

$$= \frac{S}{v}G(\alpha, \beta, t), \tag{37}$$

where

$$G(\alpha, \beta, t) = \frac{1}{\sqrt{t_p}}\left\{\frac{\alpha E(\alpha, t)}{|\bar\beta - \alpha|^2}\right.$$

$$\left. - \frac{2}{\Im(\beta)/j}\left(\frac{\beta E(\beta, t)}{\beta - \alpha} - \frac{\bar\beta E(\bar\beta, t)t}{\bar\beta - \alpha}\right)\right\}, \tag{38}$$

$G(\alpha, \beta, t)$ is a real-valued function despite including complex values within its definition.

$\Im(\beta)$ is the imaginary part of β, and $\bar\beta$ is the complex conjugate of β while the function $E(\varphi, t)$ is

$$E(\varphi, t) = (1 + \varphi\sqrt{t_p})(\mathrm{erfc}(-\varphi\sqrt{t}))\exp\varphi^2 t$$

$$= (1 + \varphi\sqrt{t_p})(1 + \mathrm{erf}\,\varphi\sqrt{t})\exp\varphi^2 t, \tag{39}$$

where $\mathrm{erf}(x)$ is the error function.

Likewise the response of the building volume air to a pulse of external vapour pressure of amplitude Δp_o, is found by taking the inverse Laplace Transform of the expression in Eq. (35) and gives

$$\Delta p_i = \Delta p_o F G(\alpha, \beta, t). \tag{40}$$

4.2. Vapour pressure response to an impulse of temperature

Taking the inverse Laplace Transform of the expression in Eq. (36) and multiplying by ΔT_i gives the response of the vapour pressure of the building volume air to a pulse of the building volume temperature of amplitude ΔT_i as

$$\Delta p_i = \Delta T_i H(\alpha, \beta, t), \tag{41}$$

where

$H(\alpha, \beta, t)$

$$= -\frac{\frac{\partial m}{\partial T}\frac{A}{v}}{\sqrt{t_T t_p}\left(\sqrt{\frac{1}{\kappa}} + \sqrt{\frac{\partial m/\partial p}{D}}\right)}$$

$$\left\{\begin{array}{l}\dfrac{1/\sqrt{t_T}J(-1/\sqrt{t_T})}{(-1/\sqrt{t_T} - \alpha)(-\bar\beta - 1/\sqrt{t_T})(1/\sqrt{t_T} + \beta)} \\[2mm] +\dfrac{\alpha J(\alpha)}{(\alpha - \beta)(-1/\sqrt{t_T} - \alpha)(-\alpha + \beta)} \\[2mm] +\dfrac{\beta J(\beta)}{(1/\sqrt{t_T} + \beta)(\beta - \bar\beta)(-\alpha + \beta)} \\[2mm] -\dfrac{\bar\beta J(\bar\beta)}{(\alpha - \bar\beta)(-\beta - 1/\sqrt{t_T})(-\beta - \bar\beta)}\end{array}\right\} \tag{42}$$

and where

$$J(\varphi) = \exp(\varphi^2 t)\,\mathrm{erfc}(-\varphi\sqrt{t}). \tag{43}$$

This vapour pressure response to a temperature excitation is very small but is included for completeness. Fig. 1 shows that a temperature impulse of magnitude $1°C$ gives rise to a transient peak in building volume vapour pressure of just over 0.11 Pa just before 1000 s have elapsed, under the conditions: $R = 0.1°C\ m^2\ W^{-1}$, $\kappa = 10^{-6}\ m^2\ s^{-1}$, $k = 0.1\ W\ m^{-1}\ °C^{-1}$, $D = 10^{-11}\ s$, $\partial m/\partial p = 0.1\ kg\ m^{-3}\ Pa^{-1}$, $\partial m/\partial T = -1\ kg\ m^{-3}\ °C^{-1}$, $r = 10^{-8}\ N\ s\ kg^{-1}$, $v = 10^{-5}\ kg\ Pa^{-1}$.
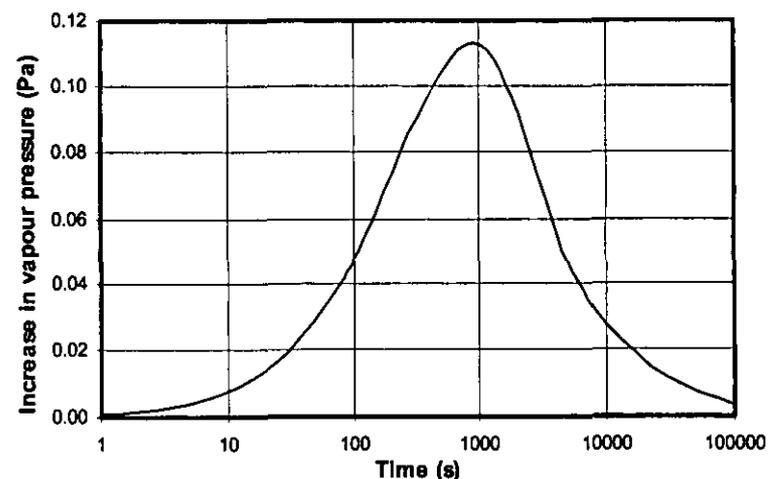

Fig. 1. The effect of a unit pulse of temperature on the vapour pressure of the building volume, from Eq. (42) ($R=0.1°C\,m^2\,W^{-1}$, $\kappa=10^{-6}\,m^2\,s^{-1}$, $k = 0.1\,W\,m^{-1}\,°C^{-1}$, $D = 10^{-11}\,s$, $\partial m/\partial p = 0.1\,kg\,m^{-3}\,Pa^{-1}$, $\partial m/\partial T = -1\,kg\,m^{-3}\,°C^{-1}$, $r = 10^{-8}\,N\,s\,kg^{-1}$, $v = 10^{-5}\,kg\,Pa^{-1}$).

The vapour pressure responses to a temperature step function and to a sinusoidal temperature excitation have been derived but, since the response is very small, they are not reported here.

4.3. Vapour pressure response to a vapour pressure step function excitation

Recall that the Laplace Transform for a step of moisture release of magnitude S is $\mathbf{S} = S/s$. This expression is substituted into Eq. (37) and then the inverse Laplace Transform

$$\mathscr{L}^{-1}\left(\frac{1}{s(\sqrt{s}-a)}\right) = -\frac{1}{a} + \frac{1}{a}\exp(a^2 t)\operatorname{erfc}(-a\sqrt{t}) \quad (44)$$

is utilised which yields, after simplification

$$\Delta p_i = \left(\frac{S}{vF}\right)\left(1 - \frac{F}{\sqrt{t_p}}\left(\frac{E(\alpha,t)}{\alpha(\alpha-\beta)(-\alpha+\bar{\beta})}\right.\right.$$
$$\left.\left. - \frac{E(\beta,t)}{\beta(-\alpha+\beta)(\beta-\bar{\beta})} + \frac{E(\bar{\beta},t)}{\bar{\beta}(-\alpha+\bar{\beta})(\beta-\bar{\beta})}\right)\right).$$
$$(45)$$

Similarly, for a step of external vapour pressure of amplitude Δp_o we find

$$\Delta p_i = \Delta p_o\left\{1 - \frac{F}{\sqrt{t_p}}\left(\frac{E(\alpha,t)}{\alpha(\alpha-\beta)(-\alpha+\bar{\beta})}\right.\right.$$
$$\left. - \frac{E(\beta,t)}{\beta(-\alpha+\beta)(\beta-\bar{\beta})}\right.$$
$$\left.\left. + \frac{E(\bar{\beta},t)}{\bar{\beta}(-\alpha+\bar{\beta})(\beta-\bar{\beta})}\right)\right\}. \quad (46)$$

The vapour pressure response to a vapour pressure sinusoidal excitation is found to be

$$\Delta p_i = \Delta p_o \frac{\sqrt{1 + \sqrt{2\omega t_p} + \omega t_p}}{M(\omega)}\exp(j\omega t - \theta + \phi), \quad (47)$$

where

$$M(\omega) = \sqrt{1 + \left(\frac{\omega}{F}\right)^2 + \left(1 + \frac{1}{Ft_h} - \frac{\omega}{F}\right)\left(1 + \frac{\omega}{F}\right)\sqrt{\omega t_p} + \left(1 + \frac{1}{Ft_h} - \frac{\omega}{F}\right)^2 \omega t_p}, \quad (48)$$

$$\phi = \tan^{-1}\left\{\frac{\sqrt{\omega t_p/2}}{1 + \sqrt{\omega t_p/2}}\right\} \quad (49)$$

and

$$\theta = \tan^{-1}\left\{\frac{\frac{\omega}{F} + \sqrt{\frac{\omega t_p}{2}}(1 + \frac{1}{Ft_h} - \frac{\omega}{F})}{1 + \sqrt{\frac{\omega t_p}{2}}(1 + \frac{1}{Ft_h} - \frac{\omega}{F})}\right\}. \quad (50)$$

5. Discussion

Eq. (37) shows that the time development of the building volume vapour pressure is governed essentially by the values of α and β in the function $G(\alpha, \beta, t)$. α and β have the units of $(\text{time})^{-1/2}$ so that $1/\alpha^2$ and $1/\beta^2$ are similar to time constants (although the shape of $G(\alpha, \beta, t)$ is not exponential). The values of α and β from Eq. (33) are determined as the real and one of the complex conjugate roots of the equation

$$x^3 + \frac{x^2}{\sqrt{t_p}} + \left(F + \frac{1}{t_h}\right)x + \frac{F}{\sqrt{t_p}} = 0. \quad (51)$$

In the limit of large air change rates, i.e. large F, Eq. (51) is approximately

$$Fx + \frac{F}{\sqrt{t_p}} \cong 0,$$

i.e. $x = \alpha \cong -1/\sqrt{t_p}$. Since $1/\alpha^2$ is approximately t_p (as defined in Eq. (32)) we have t_p as one of the characteristic times, loosely one of the characteristic time constants, describing the behaviour of the building volume vapour pressure when the air leakage is high.

By noting that the sum of the roots of Eq. (51) is $-1/\sqrt{t_p}$ and the product of the roots is $-F/\sqrt{t_p}$ it is straightforward to show that $\beta = j\sqrt{F}$ which in turn implies that the second "time constant" governing the behaviour of the building volume vapour pressure is, as expected, just $1/F$ when the air leakage is very high.

In the limit of very low air change rates, $F \cong 0$, the building volume vapour pressure is governed chiefly by the equivalent hygroscopic volume, in which case Eq. (51) is approximately

$$x^3 + \frac{x^2}{\sqrt{t_p}} + \frac{1}{t_h}x \cong 0,$$

i.e.

$$x \cong 0 \quad \text{and} \quad x \cong -1/(2\sqrt{t_p}) \pm \sqrt{1/(4t_p) - 1/t_h}.$$

For building materials t_h is of the order of 10^2 s, and t_p is of the order of 10^4 s so that

$$x \cong \pm j\sqrt{1/t_h} = \beta \quad \text{and} \quad \bar{\beta}. \quad (52)$$

Eq. (52) states that one of the characteristic times, loosely one of the characteristic time constants, describing the behaviour of the building volume vapour pressure is just t_h (as defined in Eq. (31)) when the air leakage is low $(t_h \ll t_p)$. From the definition of t_h it can be seen that under these conditions, as expected, the behaviour of the building volume vapour pressure is governed by the properties of the equivalent hygroscopic volume, viz. the area A of the

equivalent hygroscopic volume air exposed to the building volume and its surface mass transfer coefficient r.

Since the product of the roots of Eq. (51) must be $-F/\sqrt{t_p}$, it is straightforward to show that

$$\alpha \cong -\frac{Ft_h}{\sqrt{t_p}}.$$

The second "time constant", $1/\alpha^2$, is approximately equal to $t_p/(Ft_h)^2$, and governs the behaviour of the building volume vapour pressure when the air leakage is very low.

Step function responses, Eqs. (15), (45) and (46) must be used with caution because they imply that moisture can enter or leave the equivalent hygroscopic volume indefinitely, i.e. the physical reality of a finite equivalent hygroscopic volume is violated by the mathematical assumption that the amount of hygroscopic materials is effectively infinite. To use this model, it is necessary that the driving conditions fluctuate rapidly enough about a mean value that the resulting fluctuating moisture transfer in the centre of the equivalent hygroscopic volume is insignificant.

6. Comparison with simpler analytical models

If the equivalent hygroscopic volume is lumped and modelled as a single node, then a double exponential model results [14]. For a step in external vapour pressure, this rather simple model takes the form

$$\Delta p_i = A \exp(\alpha' t) + B \exp(\beta' t) + \Delta p_o,$$

where A and B depend upon initial conditions, see [14] for details.

The response to a step function is rather a sensitive test of this model as the double exponential curve shows clearly as the sum of two curves. For typical values of a room with hygroscopic linings ($R=0.1^\circ\mathrm{C}\,\mathrm{m}^2\,\mathrm{W}^{-1}$, $D=10^{-11}$ s, $\partial m/\partial p = 0.1\,\mathrm{kg}\,\mathrm{m}^{-3}\,\mathrm{Pa}^{-1}$, $r=10^{-8}\,\mathrm{N}\,\mathrm{s}\,\mathrm{kg}^{-1}$, $v=10^{-5}\,\mathrm{kg}\,\mathrm{Pa}^{-1}$) the response will be as illustrated in Fig. 2. A double exponential model, with parameters adjusted to give best fit to the model of this work, is similar to this model for the first hour or so, and compares well again after about 10 days. However, clearly the simple double exponential model has the wrong shape, and this cannot be changed by adjustment of its time constants.

7. Conclusions

A building volume containing hygroscopic materials is a central issue in building physics, describing as it does a wide range of particular cases, such as a room, a sub-floor cavity, a wall cavity, an attic, a cavity in a cathedral ceiling, etc. It has been shown that it is possible to give one set of solutions in a closed analytical form that describes the physics common to all these cases. Only modest simplifications have been made, in particular, assuming that the sorption curve for the hygroscopic materials is linear over
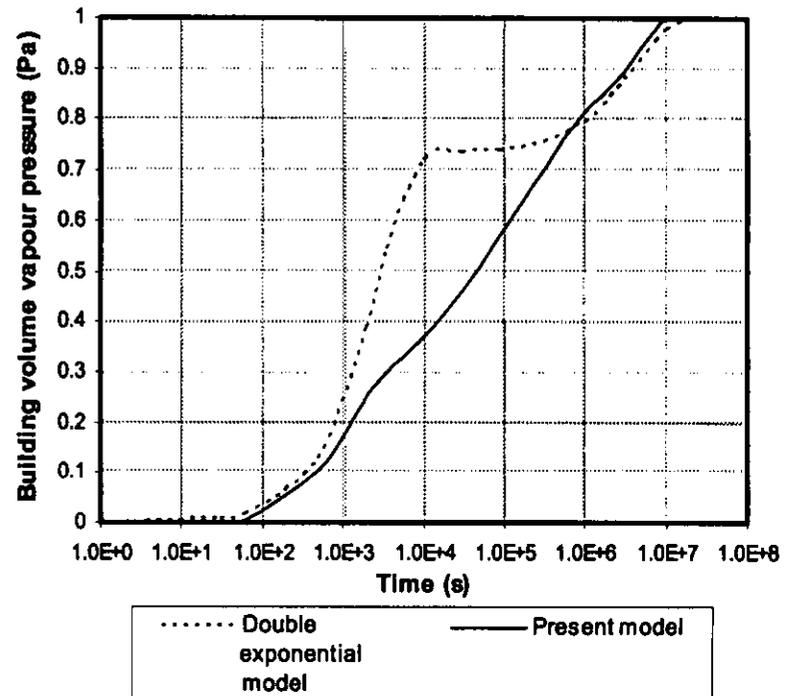

Fig. 2. A comparison of the response of the model developed in this work to that of a double exponential model of a 1 Pa step function. The conditions for model developed in this work are: $R = 0.1^\circ\mathrm{C}\,\mathrm{m}^2\,\mathrm{W}^{-1}$, $D = 10^{-11}$ s, $\partial m/\partial p = 0.1\,\mathrm{kg}\,\mathrm{m}^{-3}\,\mathrm{Pa}^{-1}$, $r=10^{-8}\,\mathrm{N}\,\mathrm{s}\,\mathrm{kg}^{-1}$, $v=10^{-5}\,\mathrm{kg}\,\mathrm{Pa}^{-1}$, while the double experimental model has had its parameters adjusted to give best fit to the model of this work.

the temperature and vapour pressure ranges under consideration, and that the psychrometric properties of water vapour pressure can be approximated as constant over these ranges also. Isothermality has not been assumed.

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